High-energy Delbruck scattering at large angles

1988 J. Phys. A: Math. Gen. 21 2941

(http://iopscience.iop.org/0305-4470/21/13/017)

J. Phys. A: Math. Gen. 21 (1988) 2941-2952. Printed in the UK

High-energy Delbrück scattering at large angles

A I Milstein and R Zh Shaisultanov
Institute of Nuclear Physics, 630090, Novosibirsk, USSR

Received 14 December 1987

Abstract. An expression for the high-energy Delbrück amplitude at large scattering angles is derived. This expression is exact in the parameter $Z\alpha$. The consideration is based on the use of the relativistic electron Green function in a Coulomb field.

1. Introduction

The elastic scattering of a photon in a Coulomb field via virtual electron–positron pairs (Delbrück scattering, Delbrück (1933)) is one of the few non-linear quantum electrodynamics processes that is directly observable in experiment. The situation when $\omega \geqslant m$ (ω is the photon frequency, m is the electron mass and $\hbar = c = 1$) is most favourable (see Papatzacos and Mork 1975a, Jarlskog *et al* 1973, Kane *et al* 1986).

At present the amplitude of the process is studied in detail in the lowest-order Born approximation (see Constantini *et al* 1971, Papatzacos and Mork 1975b, Bar-Noy and Kahane 1977, Cheng *et al* 1982). The corresponding calculations have been carried out for an arbitrary value of the momentum transfer $\boldsymbol{\Delta}$ ($\boldsymbol{\Delta} = \boldsymbol{k}_2 - \boldsymbol{k}_1$; $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are the momenta of the incoming and outgoing photons, respectively; $|\boldsymbol{k}_1| = |\boldsymbol{k}_2| = \omega$). The amplitude, exact in $Z\alpha$ ($Z|e|$ is the charge of the nucleus, $\alpha = e^2 = \frac{1}{137}$ is the fine-structure constant, e is the electron charge), has been found by Cheng and Wu (1969, 1970, 1972) in the limit $\omega/m \gg 1$ with $\Delta \ll \omega$. They have solved the problem, summing in a definite approximation the Feynman diagrams with an arbitrary number of photon exchange with a Coulomb centre. It appears that the Coulomb corrections at $Z\alpha \sim 1$ drastically change the result as compared to the Born approximation (Cheng and Wu 1969, 1972).

The main contribution to the total cross section at $\omega \gg m$ comes from the momentum transfer $\Delta \sim m$, with scattering angle $\theta_0 \sim \Delta/\omega \ll 1$. The characteristic impact parameter $\rho \sim 1/\Delta$. Therefore, the value of the angular momentum $l \sim \rho\omega \sim \omega/\Delta$ proves to be large and it is possible to employ the quasiclassical approximation. The corresponding approach intended for a description of quantum electrodynamics processes in a Coulomb field at high energies has been developed by Milstein and Strakhovenko (1983a, b). In particular, the dependence of the total cross section on the charge of a Coulomb centre has been defined. The consideration has been based on the use of the quasiclassical Green function obtained from the integral representation for the electron Green function in a Coulomb field (Milstein and Strakhovenko 1982).

In the past ten years, elastic photon scattering at large angles has been studied intensively experimentally (see, for example, Rullhusen *et al* (1983), Kasten *et al* (1986) and Kane *et al* (1986)). The results of these experiments show that the Coulomb corrections to the Delbrück scattering amplitude must be taken into account.

0305-4470/88/132941+12$02.50

In the present paper, in order to study the Coulomb corrections the exact Delbrück scattering amplitude is found in the limit $\omega/m \gg 1$, $\Delta/m \gg 1$. At $\Delta \sim \omega$ the scattering angle θ_0 ($\sin(\frac{1}{2}\theta_0) = \Delta/2\omega$) is not small and the characteristic angular momentum $l \sim 1$. Therefore in this case the quasiclassical approach is not valid and one has to develop another approach. At $m \ll \Delta \ll \omega$ the amplitude obtained agrees with the results of Cheng and Wu (1972), and Milstein and Strakhovenko (1983b). It has been shown by Cheng *et al* (1982) that the Delbrück scattering amplitude scales in the form $f(\theta_0)/\omega$ as $\omega/m \to \infty$ with θ_0 fixed. Our explicit calculation confirms this statement.

2. Calculation of the Delbrück amplitude

Let an incoming photon produce at the point $\boldsymbol{r}_1$ a pair of virtual particles that is transformed at the point $\boldsymbol{r}_2$ into an outgoing photon. In the Furry representation, the corresponding amplitude is

$$M = 2\mathrm{i}\alpha \int \mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\boldsymbol{r}_2 \exp[\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r}_1 - \boldsymbol{k}_2\cdot\boldsymbol{r}_2)] \int \mathrm{d}\varepsilon_1\,\mathrm{d}\varepsilon_2\,\delta(\omega - \varepsilon_1 + \varepsilon_2) \times \mathrm{Tr}\,\hat{e}_1 G(\boldsymbol{r}_1, \boldsymbol{r}_2|\varepsilon_2)\hat{e}_2^* G(\boldsymbol{r}_2, \boldsymbol{r}_1|\varepsilon_1) \tag{1}$$

where e_1 and e_2 are the photon polarisation vectors, $\hat{e} = e_\mu\gamma^\mu$, γ^μ are Dirac matrices and $G(\boldsymbol{r}_1, \boldsymbol{r}_2|\varepsilon)$ is the electron Green function in a Coulomb field. As is known, the function $G(\boldsymbol{r}_1, \boldsymbol{r}_2|\varepsilon)$ has, in the complex plane ε, cuts along the real axis from $-\infty$ to $-m$ and from m to ∞, which correspond to the continuous spectrum. It also has simple poles, corresponding to a discrete spectrum, in the interval $(0, m)$ for an attractive field under consideration. According to the Feynman rules, $G(\varepsilon)$ is equal to $G(\varepsilon + \mathrm{i}0)$ at $\varepsilon > 0$ and is equal to $G(\varepsilon - \mathrm{i}0)$ at $\varepsilon < 0$. The integral representation for the electron Green function in a Coulomb field which is valid in the whole complex plane ε has been obtained by Milstein and Strakhovenko (1982). Let us represent the δ function in (1) in the form

$$\delta(\omega - \varepsilon_1 + \varepsilon_2) = \frac{\mathrm{i}}{2\pi}\left(\frac{1}{\omega - \varepsilon_1 + \varepsilon_2 + \mathrm{i}0} - \frac{1}{\omega - \varepsilon_1 + \varepsilon_2 - \mathrm{i}0}\right). \tag{2}$$

Using the analytical properties of the function G, it is possible to deform the contour of the integration with respect to ε_1 and ε_2 in (1) in such a way that, with the first term in (2), the integrals with respect to ε_1 and ε_2 encircle the right- and left-hand cuts, respectively. With the second term in (2), the contours of integration with respect to ε_1 and ε_2 will encircle respectively the left- and right-hand cuts. The contribution of the discrete spectrum can be neglected when $\omega \gg m$. Performing the stated transformations, we obtain

$$M = \frac{\alpha}{\pi}\int \mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\boldsymbol{r}_2 \exp[\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r}_1 - \boldsymbol{k}_2\cdot\boldsymbol{r}_2)] \times \int_0^\infty\int_0^\infty \mathrm{d}\varepsilon_1\,\mathrm{d}\varepsilon_2\,\mathrm{Tr}\left(\frac{\hat{e}_1\delta G(\boldsymbol{r}_1, \boldsymbol{r}_2|-\varepsilon_2)\hat{e}_2^*\delta G(\boldsymbol{r}_2, \boldsymbol{r}_1|\varepsilon_1)}{\omega - \varepsilon_1 - \varepsilon_2 + \mathrm{i}0} - \frac{\hat{e}_1\delta G(\boldsymbol{r}_1, \boldsymbol{r}_2|\varepsilon_2)\hat{e}_2^*\delta G(\boldsymbol{r}_2, \boldsymbol{r}_1|-\varepsilon_1)}{\omega + \varepsilon_1 + \varepsilon_2 - \mathrm{i}0}\right) \tag{3}$$

where $\delta G(\varepsilon)=G(\varepsilon+\mathrm{i}0)-G(\varepsilon-\mathrm{i}0)$ is the discontinuity of the Green function on the cut. Note that each term in equation (3) corresponds to the contribution of the non-covariant perturbation theory diagram. Using equations (19)-(21) of Milstein and Strakhovenko (1982), we have for the function δG at $m=0$

$$\begin{aligned}\delta G(r_2, r_1|\pm|\varepsilon|) &= -\frac{\mathrm{i}}{4\pi r_1 r_2}\sum_{l=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}s \exp\{\mathrm{i}[\pm 2Z\alpha s+|\varepsilon|(r_1+r_2)\coth(s)-\pi\nu]\} \\ &\times\Big[\gamma^0[1+\boldsymbol{n}_1\cdot\boldsymbol{n}_2+\mathrm{i}\boldsymbol{\Sigma}\cdot[\boldsymbol{n}_2\times\boldsymbol{n}_1]] \\ &\times(\tfrac{1}{2}yJ'_{2\nu}(y)\mp\mathrm{i}Z\alpha\coth(s)J_{2\nu}(y))B+\gamma^0[1-\boldsymbol{n}_1\cdot\boldsymbol{n}_2+\mathrm{i}\boldsymbol{\Sigma}\cdot[\boldsymbol{n}_1\times\boldsymbol{n}_2]]AJ_{2\nu}(y) \\ &\pm\left(\frac{\mathrm{i}|\varepsilon|(r_2-r_1)}{\sinh^2(s)}\boldsymbol{\gamma}\cdot(\boldsymbol{n}_1+\boldsymbol{n}_2)B-\coth(s)\boldsymbol{\gamma}\cdot(\boldsymbol{n}_2-\boldsymbol{n}_1)A\right)J_{2\nu}(y)\Big]\end{aligned} \tag{4}$$

where $\nu=(l^2-(Z\alpha)^2)^{1/2}$, $x=\boldsymbol{n}_1\cdot\boldsymbol{n}_2$, $B=(\mathrm{d}/\mathrm{d}x)(P_l(x)-P_{l-1}(x))$, $A=l(\mathrm{d}/\mathrm{d}x)(P_l(x)+P_{l-1}(x))$, $y=2|\varepsilon|\sqrt{r_1r_2}/\sinh(s)$, $\boldsymbol{n}_i=\boldsymbol{r}_i/r_i$ $(i=1,2)$. In formula (4) $J_{2\nu}(y)$ are Bessel functions, $P_l(x)$ are Legendre polynomials, and $J'_{2\nu}(y)=(\mathrm{d}/\mathrm{d}y)J_{2\nu}(y)$.

Let us make the following change of variables: $r_1=Rt$, $r_2=R/t$, $\varepsilon_{1,2}R=p_{1,2}$. It is easy to see that the substitution $\boldsymbol{r}_i\to-\boldsymbol{r}_i$ does not change the trace in (3). Using this fact, one can show that the sum of two terms in (3) is equal to the first term, where the integration over R is extended from $-\infty$ to ∞. In consequence, we have the following integral over R:

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}R\cos(\omega Ra)}{\omega R-p_1-p_2+\mathrm{i}0}=-\frac{\mathrm{i}\pi}{\omega}\exp[-\mathrm{i}|a|(p_1+p_2)] \tag{5}$$

where $a=\boldsymbol{\lambda}_1\cdot\boldsymbol{n}_1t-\boldsymbol{\lambda}_2\cdot\boldsymbol{n}_2/t$, $\boldsymbol{\lambda}_{1,2}=\boldsymbol{k}_{1,2}/\omega$. One can see from (5) and (6) that we have now factorised the integrals with respect to the variables p_1, p_2, s_1, s_2.

Let us consider a typical integral (other integrals can be calculated similarly):

$$\begin{aligned}N &= \int_0^{\infty}\mathrm{d}p\int_{-\infty}^{\infty}\mathrm{d}s\exp[\mathrm{i}(\varphi-p|a|)]J_{2\nu}\left(\frac{2p}{\sinh(s)}\right) \\ &= \int_0^{\infty}\mathrm{d}p\int_0^{\infty}\mathrm{d}s\exp(-\mathrm{i}p|a|)J_{2\nu}\left(\frac{2p}{\sinh(s)}\right)(\mathrm{e}^{\mathrm{i}\varphi}+\mathrm{e}^{-\mathrm{i}\varphi}) \\ \varphi &= 2Z\alpha s+p(t+1/t)\coth(s)-\pi\nu\end{aligned} \tag{6}$$

where the relation $J_{2\nu}(\mathrm{e}^{\mathrm{i}\pi}x)=\mathrm{e}^{2\mathrm{i}\pi\nu}J_{2\nu}(x)$ is used. Next we change over to the variable $p/\sinh(s)\to p$ and deform the contour of the integration over p in the second term so that the integral is extended from 0 to $-\infty$ ($p\to p\,\mathrm{e}^{-\mathrm{i}\pi}$). As a result, we have

$$\begin{aligned}N &= \mathrm{e}^{-\mathrm{i}\pi\nu}\int_0^{\infty}\mathrm{d}p\,J_{2\nu}(2p)\int_{-\infty}^{\infty}\mathrm{d}s\sinh(s) \\ &\times\exp\{\mathrm{i}[p(t+1/t)\cosh(s)+2Z\alpha s-p|a|\sinh(s)]\} \\ &= \pi\exp[\mu(s_0+\tfrac{1}{2}\mathrm{i}\pi)-\mathrm{i}\pi\nu]\int_0^{\infty}\mathrm{d}p\,J_{2\nu}(2p) \\ &\times[\sinh(s_0)\dot{H}_{\mu}^{(1)}(p\rho)-(\mu/p\rho)\cosh(s_0)H_{\mu}^{(1)}(p\rho)]\end{aligned} \tag{7}$$

where $\mu = 2iZ\alpha$, $H_\mu^{(1)}(x)$ is the Hankel function of the first kind, $\dot{H}_\mu^{(1)}(x) = (d/dx)H_\mu^{(1)}(x)$, $\rho = [(t+1/t)^2 - a^2]^{1/2}$, $\sinh(s_0) = |a|/\rho$, $\cosh(s_0) = (t+1/t)/\rho$. Calculating the integral over s in (7) we have made the change of variable $s \to s + s_0$, and we have used the standard definition of the Hankel functions. Taking the integrals over s_1 and s_2, as in deriving equation (7), we get for the amplitude M (8):

$$M = -\frac{i\alpha}{2\omega}\sum_{l_1,l_2=1}^{\infty}\int_0^\infty \frac{dt}{t}\int dn_1\, dn_2\{\tfrac{1}{2}\Phi_{\nu_1}(\rho)\Phi_{\nu_2}(\rho)B_1B_2[Z_1a^2 + Z_4(t-1/t)^2]$$
$$+ 2A_1A_2[\dot{F}_{\nu_1}(\rho)\dot{F}_{\nu_2}(\rho)(Z_2\sinh^2(s_0) + Z_5\cosh^2(s_0))$$
$$-(\mu^2/\rho^2)F_{\nu_1}(\rho)F_{\nu_2}(\rho)(Z_2\cosh^2(s_0) + Z_5\sinh^2(s_0))]$$
$$-2A_1B_2\dot{F}_{\nu_1}(\rho)\Phi_{\nu_2}(\rho)[|a|Z_3\sinh(s_0) + (t-1/t)Z_6\cosh(s_0)]\} \tag{8}$$

where

$$\Phi_\nu(\rho) = \exp[i\pi(\tfrac{1}{2}\mu - \nu)]\int_0^\infty p\, dp\, J_{2\nu}(2p)H_\mu^{(1)}(\rho p)$$
$$F_\nu(\rho) = \exp[i\pi(\tfrac{1}{2}\mu - \nu)]\int_0^\infty \frac{dp}{p} J_{2\nu}(2p)H_\mu^{(1)}(\rho p). \tag{9}$$

The functions $\Phi_\nu(\rho)$ and $F_\nu(\rho)$ are expressed via hypergeometric functions (see the appendix). The subscripts 1, 2 in A_1, B_1, ν_1, A_2, B_2, ν_2 denote the dependence of these quantities on l_1 and l_2, respectively (for the definition see after equation (4)). The coefficients Z_i are

$$Z_1 = (\boldsymbol{n}_1\cdot\boldsymbol{n}_2)(1+\boldsymbol{n}_1\cdot\boldsymbol{n}_2)(\boldsymbol{e}_1\cdot\boldsymbol{e}_2^*) + (\boldsymbol{e}_1\cdot[\boldsymbol{n}_1\times\boldsymbol{n}_2])(\boldsymbol{e}_2^*\cdot[\boldsymbol{n}_1\times\boldsymbol{n}_2])$$
$$+(1+\boldsymbol{n}_1\cdot\boldsymbol{n}_2)[\boldsymbol{e}_2^*\times\boldsymbol{e}_1]\cdot[\boldsymbol{n}_2\times\boldsymbol{n}_1]$$
$$Z_4 = (\boldsymbol{e}_1\cdot(\boldsymbol{n}_1+\boldsymbol{n}_2))(\boldsymbol{e}_2^*\cdot(\boldsymbol{n}_1+\boldsymbol{n}_2)) - (\boldsymbol{e}_1\cdot\boldsymbol{e}_2^*)(1+\boldsymbol{n}_1\cdot\boldsymbol{n}_2)$$
$$Z_6 = (\boldsymbol{e}_1\cdot\boldsymbol{n}_2)(\boldsymbol{e}_2^*\cdot\boldsymbol{n}_2) - (\boldsymbol{e}_1\cdot\boldsymbol{n}_1)(\boldsymbol{e}_2^*\cdot\boldsymbol{n}_1) \tag{10}$$
$$Z_2 = Z_1(\boldsymbol{n}_1\to-\boldsymbol{n}_1) \qquad Z_3 = \tfrac{1}{2}(Z_1+Z_2) - \boldsymbol{e}_1\cdot\boldsymbol{e}_2^*$$
$$Z_5 = Z_4(\boldsymbol{n}_1\to-\boldsymbol{n}_1).$$

These coefficients appear as a result of taking the trace.

Let us now discuss the polarisation properties of the amplitude. In terms of linear polarisations, by virtue of parity conservation, the amplitude differs from zero only if the polarisations of the incoming and outgoing photons both lie in the scattering plane ($M_\parallel$) or are perpendicular to it ($M_\perp$). The corresponding polarisation vectors are

$$\boldsymbol{e}_{1\parallel} = (\boldsymbol{\lambda}_2 - \beta\boldsymbol{\lambda}_1)/(1-\beta^2)^{1/2} \qquad \boldsymbol{e}_{2\parallel} = (\beta\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1)/(1-\beta^2)^{1/2}$$
$$\boldsymbol{e}_{1\perp} = \boldsymbol{e}_{2\perp} = [\boldsymbol{\lambda}_1\times\boldsymbol{\lambda}_2]/(1-\beta^2)^{1/2} \qquad \boldsymbol{e}_\perp\cdot\boldsymbol{e}_\parallel = 0 \qquad \boldsymbol{e}_{1\parallel}\cdot\boldsymbol{e}_{2\parallel} = \beta \tag{11}$$

where $\beta = \boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2$. Therefore the tensor $T^{ij} = e_1^ie_2^j$ is of the form

$$T_\perp^{ij} = \delta^{ij} + \frac{\beta}{1-\beta^2}(\lambda_1^i\lambda_2^j + \lambda_2^i\lambda_1^j) - \frac{1}{1-\beta^2}(\lambda_1^i\lambda_1^j + \lambda_2^i\lambda_2^j)$$
$$T_\parallel^{ij} = \lambda_1^i\lambda_2^j - \frac{1}{1-\beta^2}(\lambda_1^i\lambda_2^j + \lambda_2^i\lambda_1^j) + \frac{\beta}{1-\beta^2}(\lambda_1^i\lambda_1^j + \lambda_2^i\lambda_2^j). \tag{12}$$

For helical amplitudes, the following relations hold: $M_{++} = M_{--} = \tfrac{1}{2}(M_\parallel + M_\perp)$, $M_{+-} = M_{-+} = \tfrac{1}{2}(M_\parallel - M_\perp)$; the helical polarisations are defined as $\boldsymbol{e}_{1,2}^\mp = (\boldsymbol{\xi}\times\boldsymbol{\lambda}_{1,2} \pm i\boldsymbol{\xi})/\sqrt{2}$, where $\boldsymbol{\xi} = \boldsymbol{\lambda}_1\times\boldsymbol{\lambda}_2/(1-\beta^2)^{1/2}$.

Note that the amplitude M is a function of $\boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2$ when all the integrals are taken. Therefore one can use a very convenient trick: let us multiply both sides of (8) by $\delta(\boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2 - \beta)/8\pi^2$ and take the integrals over the angles of unit vectors $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$, using the relation $\iint \mathrm{d}\boldsymbol{\lambda}_1\, \mathrm{d}\boldsymbol{\lambda}_2\, \delta(\boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2 - \beta)/8\pi^2 = 1$. After doing this, the integrand for M in (8) will depend on $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ only in the combination $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2$. So the integration over $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ reduces to an integration over $x = \boldsymbol{n}_1 \cdot \boldsymbol{n}_2$.

Let us now consider the integral

$$g = \frac{1}{8\pi^2} \int \mathrm{d}\boldsymbol{\lambda}_1\, \mathrm{d}\boldsymbol{\lambda}_2\, \delta(\boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2 - \beta) \exp[\mathrm{i}(\boldsymbol{q}_1 \cdot \boldsymbol{\lambda}_1 - \boldsymbol{q}_2 \cdot \boldsymbol{\lambda}_2)]. \tag{13}$$

Using the well known expansion for a plane wave in spherical harmonics, we get

$$g = \sum_{l=0}^{\infty} (2l+1) j_l(q_1) j_l(q_2) p_l(\beta) p_l(\psi) \tag{14}$$

where $j_l(x) = (\pi/2x)^{1/2} J_{l+1/2}(x)$ is the spherical Bessel function, and ψ is the angle between the vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. In order to take the integral

$$g^{ij} = \frac{1}{8\pi^2} \int \mathrm{d}\boldsymbol{\lambda}_1\, \mathrm{d}\boldsymbol{\lambda}_2\, \delta(\boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2 - \beta) \exp[\mathrm{i}(\boldsymbol{q}_1 \cdot \boldsymbol{\lambda}_1 - \boldsymbol{q}_2 \cdot \boldsymbol{\lambda}_2)] T^{ij} \tag{15}$$

one can replace $\boldsymbol{\lambda}_1$ by $-\mathrm{i}\boldsymbol{\nabla}_{q_1}$ and $\boldsymbol{\lambda}_2$ by $\mathrm{i}\boldsymbol{\nabla}_{q_2}$ in T^{ij} (12), and act on g (13) by the operator obtained. We shall illustrate our further calculations by the consideration of a typical integral:

$$G = \int_0^{\infty} \frac{\mathrm{d}t}{t} \int \frac{\mathrm{d}\boldsymbol{\lambda}_1\, \mathrm{d}\boldsymbol{\lambda}_2}{8\pi^2} \delta(\boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2 - \beta) \mathscr{D}(\rho) \tag{16}$$

where $\mathscr{D}(\rho)$ is a function of the variable $\rho = [(t+1/t)^2 - a^2]^{1/2}$, $a = \boldsymbol{\lambda}_1 \cdot \boldsymbol{n}_1 t - \boldsymbol{\lambda}_2 \cdot \boldsymbol{n}_2/t$ (see (7)).

Making the identical transformation

$$\mathscr{D}(\rho) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathscr{D}([(t+1/t)^2 - x^2]^{1/2}) \int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta}{2\pi} \exp[-\mathrm{i}\zeta(x + \boldsymbol{\lambda}_1 \cdot \boldsymbol{n}_1 t - \boldsymbol{\lambda}_2 \cdot \boldsymbol{n}_2/t)] \tag{17}$$

substituting (17) into (16) and using (13) we obtain

$$G = 2 \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} \frac{\mathrm{d}t}{t} \int_0^{\infty} \mathrm{d}x\, \mathscr{D}([(t+1/t)^2 - x^2]^{1/2})$$
$$\times \int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta}{2\pi} \cos(\zeta x) j_l(\rho t) j_l(\zeta/t) P_l(\beta) P_l(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2). \tag{18}$$

Let us use now the following formula for a product $j_l(x_1) j_l(x_2)$ (Gradstein and Ryzhik (1963), p 838):

$$j_l(x_1) j_l(x_2) = \tfrac{1}{2} \int_{-1}^{1} \mathrm{d}y\, P_l(y) \frac{\sin(\varkappa)}{\varkappa}$$
$$= \tfrac{1}{2} \int_{-1}^{1} \mathrm{d}y\, P_l(y) \int_0^1 \mathrm{d}z \cos(\varkappa z) \tag{19}$$
$$\varkappa = (x_1^2 + x_2^2 - 2x_1 x_2 y)^{1/2}.$$

With (19), we take the integral over ζ and then over x. We obtain

$$G=\frac{1}{2}\sum_{l=0}^{\infty}(2l+1)P_l(\beta)P_l(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)\int_0^{\infty}\frac{\mathrm{d}t}{t}\int_0^1\mathrm{d}z\int_{-1}^{1}\mathrm{d}y\,P_l(y)$$
$$\times\mathscr{D}([(t+1/t)^2-z^2(t^2+1/t^2-2y)]^{1/2}). \qquad (20)$$

Then we make a transformation as in equation (17):

$$\mathscr{D}=\int_0^{\infty}\rho\,\mathrm{d}\rho\,\mathscr{D}(\rho)\int_{-\infty}^{\infty}\frac{\mathrm{d}\zeta}{2\pi}\exp\{\mathrm{i}\zeta[\tfrac{1}{2}(t^2+1/t^2)(1-z^2)+yz^2+1-\tfrac{1}{2}\rho^2]\}. \qquad (21)$$

Substituting (21) into (20) and making the change of variable $t=\mathrm{e}^{\varphi/2}$, we carry out the integration over φ and y. We obtain

$$G=\frac{1}{4}\sum_{l=0}^{\infty}(2l+1)P_l(\beta)P_l(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)\int_0^{\infty}\rho\,\mathrm{d}\rho\,\mathscr{D}(\rho)\int_0^1\mathrm{d}z\int_0^{\infty}\mathrm{d}\zeta\,j_l(\zeta z^2)$$
$$\times\left(i^{l+1}\exp[\mathrm{i}\zeta(1-\tfrac{1}{2}\rho^2)]H_0^{(1)}(\zeta(1-z^2))\right.$$
$$\left.+\frac{1}{i^{l+1}}\exp[-\mathrm{i}\zeta(1-\tfrac{1}{2}\rho^2)]H_0^{(2)}(\zeta(1-z^2))\right). \qquad (22)$$

Here $H_0^{(1)}$ and $H_0^{(2)}$ are the Hankel functions of the first and the second kind, respectively. One can now take the integral first over z and then over ζ, using the relations (Gradstein and Ryzhik (1963), pp 692, 725)

$$\int_0^1\frac{\mathrm{d}z}{z}J_\mu(\zeta z)J_\nu(\zeta(1-z))=\frac{J_{\mu+\nu}(\zeta)}{\mu}$$
$$\int_0^{\infty}\frac{\mathrm{d}x}{\sqrt{x}}\mathrm{e}^{\mathrm{i}px}J_{\nu+1/2}(x)=(2/\pi)^{1/2}\exp[\tfrac{1}{2}\mathrm{i}\pi(\nu+1)]Q_\nu(p+\mathrm{i}0) \qquad (23)$$

where Q_ν are the Legendre functions. The representation of the Hankel function

$$H_0^{(1,2)}(x)=J_0(x)\pm(2\mathrm{i}/\pi)\tilde{J}_0(x) \qquad \tilde{J}_0(x)\equiv(\partial/\partial\nu)J_\nu(x)|_{\nu=0}$$

has to be used as well. Finally, one obtains, for the integral G

$$G=\sum_{l=0}^{\infty}P_l(\beta)P_l(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)\int_0^{\infty}\rho\,\mathrm{d}\rho\,\mathscr{D}(\rho)S_1(\rho) \qquad (24)$$

where $S_1(\rho)$ is

$$S_1(\rho)=\tfrac{1}{2}(-1)^{l+1}\vartheta(2-\rho)\tilde{P}_l(1-\tfrac{1}{2}\rho^2)+\vartheta(\rho-2)Q_l(\tfrac{1}{2}\rho^2-1) \qquad (25)$$

$\tilde{P}_l(x)\equiv(\partial/\partial\nu)P_\nu(x)|_{\nu=l}$. Then it is easy to take the integrals over $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ in the amplitude M (8). The corresponding integrals are expressed via the Wigner 3-j symbols. In the same manner, one can obtain the expression for the whole amplitude M.

As mentioned above, there are two amplitudes $M_\parallel$ and $M_\perp$ which differ from zero, in terms of linear polarisations. It is convenient to consider the combination $M_1=\frac{1}{2}(M_\parallel+\beta M_\perp)$ and the amplitude $M_2\equiv M_\perp$. Let us start with the evaluation of M_1. After the integration over λ_1 and λ_2, we obtain that the coefficients (10) in equation (8) for the amplitude M_1 should be replaced by

$$Z_i=\frac{1}{2}\sum_{l=0}^{\infty}(2l+1)P_l(\beta)f_i$$

$$f_{1,4}=\pm\left[(1+x)P_l(x)\left(\beta j_l(q_1)j_l(q_2)+j_l'(q_1)j_l'(q_2)\pm l(l+1)\frac{j_l(q_1)j_l(q_2)}{q_1q_2}\right)\right.$$
$$\left.-(1-x^2)P_l'(x)\frac{j_l(q_1)j_l(q_2)}{q_1q_2}\right]$$
$$f_{2,5}=\pm\left[(1-x)P_l(x)\left(\beta j_l(q_1)j_l(q_2)-j_l'(q_1)j_l'(q_2)\mp l(l+1)\frac{j_l(q_1)j_l(q_2)}{q_1q_2}\right)\right.$$
$$\left.-(1-x^2)P_l'(x)\frac{j_l(q_1)j_l(q_2)}{q_1q_2}\right] \tag{26}$$
$$f_3=-(1-x^2)P_l'(x)\left(\frac{2j_l(q_1)j_l(q_2)}{q_1q_2}+j_l'(q_1)j_l(q_2)/q_2+j_l'(q_2)j_l(q_1)/q_1\right)$$
$$f_6=-(1-x^2)P_l'(x)[j_l'(q_2)j_l(q_1)/q_1-j_l'(q_1)j_l(q_2)/q_2]$$

where $x=\boldsymbol{n}_1\cdot\boldsymbol{n}_2$, $j_l'(x)=(\mathrm{d}/\mathrm{d}x)j_l(x)$, $q_1=\zeta t$, $q_2=\zeta/t$. The representations for the products of Bessel functions in (26) are given in the appendix. Then we take the integrals in the sequence used in deriving equation (24). We represent our amplitudes $M_{1,2}$ as follows:

$$M=-\frac{8\mathrm{i}\alpha\pi^2}{\omega}\sum_{l_1,l_2=1}^{\infty}\sum_{l=0}^{\infty}\int_0^{\infty}\rho\,\mathrm{d}\rho\langle\Phi_{\nu_1}(\rho)\Phi_{\nu_2}(\rho)C_1$$
$$-[\dot F_{\nu_1}(\rho)\dot F_{\nu_2}(\rho)+(\mu^2/\rho^2)F_{\nu_1}(\rho)F_{\nu_2}(\rho)]C_2$$
$$-[\dot F_{\nu_1}(\rho)\dot F_{\nu_2}(\rho)-(\mu^2/\rho^2)F_{\nu_1}(\rho)F_{\nu_2}(\rho)]C_3-(4/\rho)\dot F_{\nu_1}(\rho)\Phi_{\nu_2}(\rho)C_4\rangle. \tag{27}$$

After cumbersome calculations we obtain for the coefficients C_i in M_1 the following expressions:

$$C_1=\tfrac{1}{2}\{\gamma_1[l(l+1)(S_3-S_2)+\sigma(\beta S_1+S_5)]-\sigma\gamma_2S_2\}P_l(\beta)$$
$$C_2=\tfrac{1}{2}\{\gamma_3[l(l+1)S_2+\beta S_1-S_5]-\gamma_4S_2\}P_l(\beta)$$
$$C_3=\frac{2l(l+1)}{\rho^2}\gamma_3(S_3+\sigma S_2)P_l(\beta) \tag{28}$$
$$C_4=-\tfrac{1}{4}[3S_3+(2\sigma-1)(S_4+2S_2)+S_6]\gamma_5P_l(\beta).$$

Here $\sigma=1-\frac{1}{4}\rho^2$, $\beta=\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2=\cos\theta_0$. The coefficients γ_i are the integrals over $x=\boldsymbol{n}_1\cdot\boldsymbol{n}_2$:

$$\gamma_1=\int_{-1}^{1}\mathrm{d}x(1+x)P_l(x)B_1(x)B_2(x)\qquad\gamma_3=(-1)^{l_1+l_2+l}l_1l_2\gamma_1$$
$$\gamma_2=\int_{-1}^{1}\mathrm{d}x(1-x^2)P_l'(x)B_1(x)B_2(x)\qquad\gamma_4=(-1)^{l_1+l_2+l+1}l_1l_2\gamma_2 \tag{29}$$
$$\gamma_5=\int_{-1}^{1}\mathrm{d}x(1-x^2)P_l'(x)A_1(x)B_2(x).$$

These coefficients are expressed using Wigner 3-j symbols (see the appendix). The functions S_i appear as a result of integration over the parameters ζ and z (compare with equations (23)-(25)):

$$S_1=\tfrac{1}{2}(-1)^{l+1}\vartheta(2-\rho)\tilde P_l+\vartheta(\rho-2)Q_l$$

$$S_2=\frac{(-1)^l\vartheta(2-\rho)}{2(2l+1)}\left(\frac{\tilde{P}_{l-1}}{2l-1}+\frac{\tilde{P}_{l+1}}{2l+3}+\frac{P_{l-1}-P_{l+1}}{2l+1}\right)+\frac{\vartheta(\rho-2)}{2l+1}\left(\frac{Q_{l-1}}{2l-1}+\frac{Q_{l+1}}{2l+3}\right)$$

$$S_3=(-1)^{l+1}\vartheta(2-\rho)\left(\frac{\tilde{P}_l}{(2l-1)(2l+3)}+\frac{P_{l-2}-P_l}{4(2l-1)^2}+\frac{P_l-P_{l+2}}{4(2l+3)^2}\right)$$
$$+\left(\frac{2Q_l}{(2l-1)(2l+3)}+\frac{\delta_{1l}}{2}\right)\vartheta(\rho-2) \tag{30}$$

$$S_4=\frac{(-1)^l\vartheta(2-\rho)}{2l+1}\left(\frac{l\tilde{P}_{l-1}}{2l-1}-\frac{(l+1)\tilde{P}_{l+1}}{2l+3}+\frac{P_{l+1}-P_{l-1}}{2(2l+1)}\right)$$
$$+\vartheta(\rho-2)\left[\left(\frac{lQ_{l-1}}{2l-1}-\frac{(l+1)Q_{l-1}}{2l+3}\right)\frac{2}{2l+1}-\delta_{l0}\right]$$

$$S_5=\tfrac{1}{2}(-1)^l\vartheta(2-\rho)\left(\frac{l\tilde{P}_{l-1}}{2l-1}+\frac{(l+1)\tilde{P}_{l+1}}{2l+3}\right)+\vartheta(\rho-2)\left(\frac{lQ_{l-1}}{2l-1}+\frac{(l+1)Q_{l+1}}{2l+3}\right)$$

$$S_6=\tfrac{1}{4}(-1)^l\vartheta(2-\rho)\left(\frac{P_l-P_{l-2}}{2l-1}+\frac{P_l-P_{l+2}}{2l+3}\right)+\tfrac{1}{2}\vartheta(\rho-2)\delta_{1l}$$

where the Legendre polynomials and $\tilde{P}_l$ (recall that $\tilde{P}_l(x)=(\mathrm{d}/\mathrm{d}\nu)P_\nu(x)|_{\nu=l}$) depend on $q=1-\frac{1}{2}\rho^2$, and the Legendre functions Q_l depend on $p=\frac{1}{2}\rho^2-1$. It is easy to verify that C_i have no singularities at $\rho=2$. By virtue of the momentum conservation $M(Z=0)=0$ for the case under study, $\Delta\neq 0$. It is convenient to subtract, from the integrand of M in (27), its value at $Z=0$. This subtraction removes fictitious divergences, which cancel after summation over $l_{1,2}$ and l. In the following such a subtraction is assumed to be made.

The calculations of the coefficients C_i in M_2 are performed in a similar fashion:

$$C_1=\sigma P_l(\beta)\left(\gamma_1 S_1+\frac{2}{1-\beta^2}[\beta\gamma_2 S_2+\tfrac{1}{2}\gamma_6(S_3+S_6)-S_3\gamma_7]\right)+\gamma_1(S_3-S_2)P_l'(\beta)$$

$$C_2=P_l(\beta)\left(\gamma_1 S_1+\frac{2}{1-\beta^2}[\beta\gamma_4 S_2+\tfrac{1}{2}\gamma_8(S_3+S_6)-\gamma_9 S_3]\right)+\frac{4}{\rho^2}(1-\sigma)\gamma_3 S_2 P_l'(\beta)$$

$$C_3=(4/\rho^2)\gamma_3(\sigma S_2+S_3)P_l'(\beta) \tag{31}$$

$$C_4=\frac{P_l(\beta)}{1-\beta^2}\{\beta\gamma_5[(2\sigma-1)S_2+S_3]-\gamma_{10}[(2\sigma-1)S_3+S_2]$$
$$+\tfrac{1}{2}\gamma_{11}[(2\sigma-1)(S_3+S_6)+S_4]\}.$$

The coefficients γ_{6-11} are defined by

$$\gamma_6=\int_{-1}^{1}\mathrm{d}x(1-x^2)P_l(x)B_1(x)B_2(x)$$

$$\gamma_7=\int_{-1}^{1}\mathrm{d}x\,x(1-x^2)P_l'(x)B_1(x)B_2(x)$$

$$\gamma_8=(-1)^{l_1+l_2+l}l_1l_2\gamma_6 \qquad \gamma_9=(-1)^{l_1+l_2+l}l_1l_2\gamma_7 \tag{32}$$

$$\gamma_{10}=\int_{-1}^{1}\mathrm{d}x\,x(1-x^2)P_l'(x)A_1(x)B_2(x)$$

$$\gamma_{11}=\int_{-1}^{1}\mathrm{d}x(1-x^2)P_l(x)A_1(x)B_2(x).$$

It is seen that the amplitude M (27) has a scaling form $M = f(\theta_0)/\omega$ as $\omega/m \to \infty$ with θ_0 fixed, in agreement with the result of Cheng *et al* (1982).

Let us discuss the asymptotic form of M_1 and M_2 at $\theta_0 \ll 1$ ($\Delta \ll \omega$). In this case the main contribution to the amplitudes comes from the region $l_1 \sim l_2 \sim l \sim 1/\theta_0$, $\rho \sim \theta_0$, $1+x \sim \theta_0^2$. So one can neglect $(Z\alpha)^2$ in the quantities $\nu_{1,2} = [l_{1,2}^2 - (Z\alpha)^2]^{1/2}$. After this one can take the sum over l_1 and l_2 before the integration over x ((29) and (32)). This summation can be performed using the formulae in the appendix of Milstein and Strakhovenko (1983b). Then, replacing the Legendre polynomials by its asymptotics we obtain

$$S_1 = -S_5 \simeq \frac{(-1)^l}{2l} y J_1(y) \qquad S_3 = 2S_2 \simeq \frac{(-1)^{l+1}}{6l^5} y^3 J_3(y)$$
$$S_4 = S_6 \simeq \frac{(-1)^{l+1}}{2l^3} y^2 J_2(y) \tag{33}$$

where $y = l\rho$. We also have $P_l(\beta) \simeq J_0(l\theta_0)$ and $P_l(x) \simeq (-1)^l J_0(l\theta)$, where $x = -\cos\theta \simeq -1 + \frac{1}{2}\theta^2$. Substituting these asymptotics into (27) we see that $(-1)^l$ disappears and we can replace the summation over l by an integration. Performing this integration and then taking the integral over ρ and $p_{1,2}$ (see the definitions of Φ_ν and F_ν (9)), we obtain ultimately for the small-angle asymptotics:

$$M_1 = -\mathrm{i}\frac{8\alpha}{3\omega\theta_0^2}\left(\frac{2\pi Z\alpha}{\sinh(2\pi Z\alpha)}[1-2(Z\alpha)^2] - 1\right)$$
$$M_2 = M_1 + \frac{\mathrm{i}8\alpha}{\omega\theta_0^2}(Z\alpha)^2[Z\alpha\ \mathrm{Im}\ \psi'(1-\mathrm{i}Z\alpha) - 1] \tag{34}$$

where $\psi(x) = (\mathrm{d}/\mathrm{d}x)\ln\Gamma(x)$, $\psi'(x) = (\mathrm{d}/\mathrm{d}x)\psi(x)$. Note that $M_1 \to M_{++}$ as $\beta \to 1$. Our result (34) coincides with the results of Cheng and Wu (1972), and Milstein and Strakhovenko (1983a, b). The asymptotics (34) are imaginary quantitities, but at $\theta_0 \sim 1$ the real parts of the amplitudes (27) are not equal to zero.

3. Discussion

Equations (27), (28) and (31) solve, in general form, the problem of calculation of high-energy Delbrück amplitude at large angles. One should bear in mind that in scattering by atoms the point-charge approximation is valid if $\Delta \ll R^{-1}$, where R is the radius of the nucleus. Then one has to know the corrections of order $(m/\omega)^2$ to determine ω for which our results are applicable. This problem is very difficult, but it can be solved using the technique of the present paper. We hope that the appropriate photon energy ω is not very high.

The problem of numerical calculations with the use of (27) and comparison with the experimental data is an independent one. Note that the terms with a small $l_{1,2}$ and l give the contribution to the amplitude at $\theta_0 \sim 1$. We will discuss this problem elsewhere.

Acknowledgment

We would like to thank V N Baier for his interest in this work.

Appendix

In this appendix we discuss the properties of the functions introduced in the text. Let us consider the functions $\Phi_\nu(\rho)$ and $F_\nu(\rho)$ (see (9)). These functions are expressed via the hypergeometric functions. If $\rho > 2$, we have

$$\begin{aligned}\Phi_\nu(\rho) &= \frac{\mathrm{i}}{2\pi}\left(\frac{4}{\rho^2}\right)^{\nu+1}\frac{\Gamma(1+\nu+\frac{1}{2}\mu)\Gamma(1+\nu-\frac{1}{2}\mu)}{\Gamma(2\nu+1)} \\ &\quad\times F\left(1+\nu+\tfrac{1}{2}\mu,\,1+\nu-\tfrac{1}{2}\mu;\,2\nu+1;\,\frac{4}{\rho^2+\mathrm{i}0}\right) \\ F_\nu(\rho) &= -\frac{\mathrm{i}}{2\pi}\left(\frac{4}{\rho^2}\right)^{\nu}\frac{\Gamma(\nu+\frac{1}{2}\mu)\Gamma(\nu-\frac{1}{2}\mu)}{\Gamma(2\nu+1)} \\ &\quad\times F\left(\nu+\tfrac{1}{2}\mu,\,\nu-\tfrac{1}{2}\mu;\,2\nu+1;\,\frac{4}{\rho^2+\mathrm{i}0}\right).\end{aligned} \tag{A1}$$

If $\rho < 2$, the formulae for Φ_ν and F_ν are the analytical continuation of (A1). The function $F_\nu(\rho)$ has no singularities at $\rho = 2$: $F_\nu(2) = -\mathrm{i}[2\pi(\nu^2-\frac{1}{4}\mu^2)]^{-1} = -\mathrm{i}/2\pi l^2$ (recall that $\nu^2 = l^2+\frac{1}{4}\mu^2$). The function $\Phi_\nu(\rho)$ has a singularity as $\rho \to 2$. However, the coefficients at the divergent terms do not depend on Z:

$$\begin{aligned}\Phi_\nu(\rho) \approx \frac{\mathrm{i}}{2\pi}\bigg(&\frac{1}{\frac{1}{4}\rho^2-1+\mathrm{i}0} - \nu + l^2[\ln(\tfrac{1}{4}\rho^2-1+\mathrm{i}0) \\ &+ \psi(1+\nu+\tfrac{1}{2}\mu) + \psi(1+\nu-\tfrac{1}{2}\mu) - \psi(1) - \psi(2)]\bigg).\end{aligned} \tag{A2}$$

We give now the formulae for the products of spherical Bessel functions, which are needed for the calculations, as in the derivation of equation (22). Using (19) one can obtain that

$$\begin{aligned}\frac{j_l(q_1)j_l(q_2)}{q_1q_2} &= \frac{\delta_{l0}}{\zeta^2} - \frac{1}{4}\int_{-1}^{1}\mathrm{d}y\,P_l(y)\varkappa^2\int_0^1\mathrm{d}z(1-z)^2\cos(z\varkappa\zeta) \\ j_l'(q_1)j_l(q_2)/q_2 &+ j_l'(q_2)j_l(q_1)/q_1 \\ &= -\frac{1}{4}\int_{-1}^{1}\mathrm{d}y\,P_l(y)\varkappa^2\int_0^1\mathrm{d}z(1-z^2)\cos(z\varkappa\zeta) \\ j_l'(q_1)j_l(q_2)/q_2 &- j_l'(q_2)j_l(q_1)/q_1 \\ &= -\tfrac{1}{4}(t^2-1/t^2)\int_{-1}^{1}\mathrm{d}y\,P_l(y)\int_0^1\mathrm{d}z(1-z^2)\cos(z\varkappa\zeta).\end{aligned} \tag{A3}$$

Here $q_1 = \zeta t$, $q_2 = \zeta/t$, $\varkappa = (t^2+1/t^2-2y)^{1/2}$.

To obtain the formula for the product $j_l'(q_1)j_l'(q_2)$, one has to perform an integration by parts over ζ (see (18)). Then, with the use of the second relation in (A3) we get that this product may be replaced by

$$j_l'(q_1)j_l'(q_2) \to j_l(z_1 (j_l(q_2)\left(\tfrac{1}{2}\rho - 1 - \frac{l(l+1)}{q_1q_2}\right) + \frac{j_l(q_1)}{q_1}j_l'(q_2) + \frac{j_l(q_2)}{q_2}j_l'(q_1). \tag{A4}$$

Let us calculate now the coefficients γ_i (see (29) and (32)). As mentioned above, these quantities are expressed via the Wigner 3-j symbols. We use recursion relations for the Legendre polynomials (see e.g. Gradstein and Ryzhik (1963)):

$$(1+x)\frac{d}{dx}(P_l(x)-P_{l-1}(x)) = l(P_l(x)+P_{l-1}(x))$$

$$(1-x^2)^{1/2}\frac{d}{dx}(P_l(x)-P_{l-1}(x)) = P_l^1(x)-P_{l-1}^1(x) \qquad \text{(A5)}$$

$$P_l'(x) = \tfrac{1}{2}[P_{l+1}^2(x)+l(l+1)P_{l+1}(x)]$$

and the following formula (see Edmonds (1957, p 63):

$$\int_{-1}^{1} dx\, P_l^{|m|}(x)P_{l_1}^{|m_1|}(x)P_{l_2}^{|m_2|}(x) = 2(a_{l_1}^{m_1}a_{l_2}^{m_2}a_l^m)^{-1/2}\begin{pmatrix} l & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l & l_1 & l_2 \\ -m & m_1 & m_2 \end{pmatrix} \qquad \text{(A6)}$$

where $P_l^m(x)$ are the associated Legendre polynomials, $a_l^m = (l-|m|)!/(l+|m|)!$. The signs of m, m_1 and m_2 are chosen so that $m = m_1+m_2$. Let

$$f_l = P_l + P_{l-1},\ g_l = P_l - P_{l-1},\ f_l^1 = P_l^1 + P_{l-1}^1,\ g_l^1 = P_l^1 - P_{l-1}^1. \qquad \text{(A7)}$$

With the help of (A5) we have

$$\gamma_1 = \tfrac{1}{2}\int_{-1}^{1} dx\, P_l(x)[l_1 l_2 f_{l_1}(x) f_{l_2}(x) + g_{l_1}^1(x) g_{l_2}^1(x)]$$

$$\gamma_2 = \tfrac{1}{2}\int_{-1}^{1} dx\, g_{l_1}^1(x) g_{l_2}^1(x)[P_{l+1}^2(x)+l(l+1)P_{l+1}(x)]$$

$$\gamma_3 = (-1)^{l_1+l_2+l} l_1 l_2 \gamma_1 \qquad \gamma_4 = (-1)^{l_1+l_2+l+1} l_1 l_2 \gamma_2$$

$$\gamma_5 = \tfrac{1}{2}l_1 \int_{-1}^{1} dx\, f_{l_1}^1(x) g_{l_2}^1(x)[P_{l+1}^2(x)+l(l+1)P_{l+1}(x)]$$

$$\gamma_6 = \int_{-1}^{1} dx\, P_l(x) g_{l_1}^1(x) g_{l_2(x)}^1 \qquad \text{(A8)}$$

$$\gamma_7 = \tfrac{1}{2}\int_{-1}^{1} dx\, g_{l_1}^1(x) g_{l_2}^1(x)[P_l^2(x)+l(l+1)P_l(x)]$$

$$\gamma_8 = l_1 l_2 (-1)^{l_1+l_2+l}\gamma_6 \qquad \gamma_9 = l_1 l_2(-1)^{l_1+l_2+l}\gamma_7$$

$$\gamma_{10} = \tfrac{1}{2}l_1 \int_{-1}^{1} dx\, f_{l_1}^1(x) g_{l_2}^1(x)[l(l+1)P_l(x)+P_l^2(x)]$$

$$\gamma_{11} = l_1 \int_{-1}^{1} dx\, f_{l_1}^1(x) g_{l_2}^1(x) P_l(x).$$

Substituting (A6) into (A8), the final result can be easily obtained.

References

Bar-Noy T and Kahane S 1977 *Nucl. Phys.* A **288** 132
Cheng H, Tsai Er-C and Zhu X 1982 *Phys. Rev.* D **26** 908
Cheng H and Wu T T 1969 *Phys. Rev.* **182** 1873

—— 1970 *Phys. Rev.* D **2** 2444
—— 1972 *Phys. Rev.* D **5** 3077
Constantini V, De Tollis B and Pistoni G 1971 *Nuovo Cimento* A **2** 733
Delbrück M 1933 *Z. Phys.* **84** 144
Edmonds A R 1957 *Angular momentum in quantum mechanics* (Princeton, NJ: Princeton University Press)
Gradstein I S and Ryzhik I M 1963 *Tables of Integrals, Sums, Series and Products* (Moscow: Fizmatgiz)
Jarlskog G, Jonsson L, Prunster S, Schutz H D, Willntzki H J and Winter G G 1973 *Phys. Rev.* D **8** 3813
Kane P P, Kissel L, Pratt R H and Roy S C 1986 *Phys. Rep.* **140** 75
Kasten B, Schaupp D, Rullhusen P, Smend F and Schumacher M 1986 *Phys. Rev.* C **33** 1606
Milstein A I and Strakhovenko V M 1982 *Phys. Lett.* **90A** 447
—— 1983a *Phys. Lett.* **95A** 135
—— 1983b *Sov. Phys.-JETP* **58** 8
Papatzacos P and Mork K 1975a *Phys. Rep.* **21** 81
—— 1975b *Phys. Rev.* D **12** 206
Rullhusen P, Zurnuhl U, Smend F, Schumacher M, Borner H G and Kerr S A 1983 *Phys. Rev.* C **27** 559